

ESSENTIAL NORM AND A NEW CHARACTERIZATION OF WEIGHTED COMPOSITION OPERATORS FROM WEIGHTED BERGMAN SPACES AND HARDY SPACES INTO THE BLOCH SPACE

SONGXIAO LI, RUISHEN QIAN AND JIZHEN ZHOU

ABSTRACT. In this paper, we give some estimates for the essential norm and a new characterization for the boundedness and compactness of weighted composition operators from weighted Bergman spaces and Hardy spaces to the Bloch space.

1. INTRODUCTION

Let \mathbb{D} be the open unit disk in the complex plane \mathbb{C} and $H(\mathbb{D})$ be the space of analytic functions on \mathbb{D} . For $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space, denoted by A_α^p , is the set of all functions $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{A_\alpha^p}^p = (\alpha + 1) \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha dA(z) < \infty,$$

where dA is the normalized Lebesgue area measure in \mathbb{D} such that $A(\mathbb{D}) = 1$. The Hardy space H^p is the space consisting of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{H^p}^p = \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

The Bloch space, denoted by $\mathcal{B} = \mathcal{B}(\mathbb{D})$, is the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty.$$

Under the norm $\|f\|_{\mathcal{B}} = |f(0)| + \|f\|_{\mathcal{B}}$, the Bloch space is a Banach space. See [26] for more information of the Bloch space.

Let $v : \mathbb{D} \rightarrow \mathbb{R}_+$ be a continuous, strictly positive and bounded function. An $f \in H(\mathbb{D})$ is said to belong to the weighted space, denoted by H_v^∞ , if

$$\|f\|_v = \sup_{z \in \mathbb{D}} v(z) |f(z)| < \infty.$$

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H_v^∞ is a Banach space with the norm $\|\cdot\|_v$. The weighted v is called radial, if $v(z) = v(|z|)$ for all $z \in \mathbb{D}$. For a weight v , the associated weight \tilde{v} is defined as follows.

$$\tilde{v} = (\sup\{|f(z)| : f \in H_v^\infty, \|f\|_v \leq 1\})^{-1}, z \in \mathbb{D}.$$

When $v = v_\alpha(z) = (1 - |z|^2)^\alpha$ ($0 < \alpha < \infty$), it is easy to check that $\tilde{v}_\alpha(z) = v_\alpha(z)$. In this case, we denote H_v^∞ by $H_{v_\alpha}^\infty$ and $\|f\|_{v_\alpha} = \sup_{z \in \mathbb{D}} |f(z)|(1 - |z|^2)^\alpha$.

Let $S(\mathbb{D})$ denote the set of all analytic self-maps of \mathbb{D} . Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. For $f \in H(\mathbb{D})$, the composition operator C_φ and the multiplication operator M_u are defined by

$$(C_\varphi f)(z) = f(\varphi(z)) \text{ and } (M_u f)(z) = u(z)f(z),$$

respectively. The weighted composition operator uC_φ is defined by

$$(uC_\varphi f)(z) = u(z) \cdot f(\varphi(z)), \quad f \in H(\mathbb{D}).$$

It is clear that the weighted composition operator uC_φ is the generalization of C_φ and M_u . A basic and interesting problem concerning concrete operators (such as composition operator, multiplication operator, Volterra operator, Toeplitz operator, Hankel operator and other integral type operators) is to relate operator theoretic properties to their function theoretic properties of their symbols, which attracted a lot of attention recently, the reader can refer to [2] and [26].

It is well known that C_φ is bounded on \mathcal{B} by the Schwarz-Pick lemma for any $\varphi \in S(\mathbb{D})$. The compactness of C_φ on \mathcal{B} was studied in for example [13, 19, 21]. In [21], Wulan, Zheng and Zhu proved that, for any $\varphi \in S(\mathbb{D})$, $C_\varphi : \mathcal{B} \rightarrow \mathcal{B}$ is compact if and only if $\lim_{j \rightarrow \infty} \|\varphi^j\|_{\mathcal{B}} = 0$. This result has been generalized to Bloch-type spaces by Zhao in [25] and shows that $C_\varphi : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is compact if and only if $\lim_{j \rightarrow \infty} j^{\alpha-1} \|\varphi^j\|_{\mathcal{B}^\beta} = 0$. For some results on composition operator and related operators mapping into the Bloch space see, for example, [1, 3, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18, 22, 23, 24, 25, 27] and the related references therein.

In [7], the first author of this paper and Stević obtained a characterization of the boundedness and compactness of weighted composition operator $uC_\varphi : A_\alpha^p \rightarrow \mathcal{B}$. Among others, we proved the following result.

Theorem A. *Let $1 \leq p < \infty$, $\alpha > -1$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ such that $uC_\varphi : A_\alpha^p \rightarrow \mathcal{B}$ is bounded. Then $uC_\varphi : A_\alpha^p \rightarrow \mathcal{B}$ is compact if and only if*

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)|u'(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha)/p}} = 0 \quad \text{and} \quad \lim_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)|u(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha+p)/p}} = 0.$$

In [3], Colonna obtained a new characterization by using two families functions, among others, she obtained the following result.

Theorem B. *Let $1 \leq p < \infty$, $\alpha > -1$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ such that $uC_\varphi : A_\alpha^p \rightarrow \mathcal{B}$ is bounded. Then $uC_\varphi : A_\alpha^p \rightarrow \mathcal{B}$ is compact if and only if*

$$\lim_{|a| \rightarrow 1} \|uC_\varphi f_a\|_{\mathcal{B}} = 0 \text{ and } \lim_{|a| \rightarrow 1} \|uC_\varphi g_a\|_{\mathcal{B}} = 0,$$

where

$$f_a(z) = \frac{(1 - |a|^2)^{1+(2+\alpha)(1-1/p)}}{(1 - \bar{a}z)^{3+\alpha}}, g_a(z) = \frac{(1 - |a|^2)^{1+(2+\alpha)(1-1/p)+1/p}}{(1 - \bar{a}z)^{3+\alpha+1/p}}.$$

In [3], Colonna also obtained two characterizations for the compactness of weighted composition operator $uC_\varphi : H^p \rightarrow \mathcal{B}$.

Theorem C. *Let $1 \leq p < \infty$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ such that $uC_\varphi : H^p \rightarrow \mathcal{B}$ is bounded. Then the following statements are equivalent:*

- (a) $uC_\varphi : H^p \rightarrow \mathcal{B}$ is compact.
- (b)

$$\lim_{|a| \rightarrow 1} \|uC_\varphi p_a\|_{\mathcal{B}} = 0 \text{ and } \lim_{|a| \rightarrow 1} \|uC_\varphi q_a\|_{\mathcal{B}} = 0,$$

where

$$p_a(z) = \frac{(1 - |a|^2)^{2-1/p}}{(1 - \bar{a}z)^2}, q_a(z) = \frac{(1 - |a|^2)^2}{(1 - \bar{a}z)^{2+1/p}}.$$

- (c)

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)|u'(z)|}{(1 - |\varphi(z)|^2)^{1/p}} = 0 \quad \text{and} \quad \lim_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)|u(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{(1+p)/p}} = 0.$$

The purpose of this paper is to give some estimates for the essential norm of the operator $uC_\varphi : A_\alpha^p \rightarrow \mathcal{B}$ (as well as $uC_\varphi : H^p \rightarrow \mathcal{B}$), in particular, by using $\|uC_\varphi f_a\|_{\mathcal{B}}$ and $\|uC_\varphi g_a\|_{\mathcal{B}}$ (as well as $\|uC_\varphi p_a\|_{\mathcal{B}}$ and $\|uC_\varphi q_a\|_{\mathcal{B}}$). Moreover, we give a new characterization for the boundedness, compactness and essential norm of the operator $uC_\varphi : A_\alpha^p \rightarrow \mathcal{B}$ (as well as $uC_\varphi : H^p \rightarrow \mathcal{B}$) by using φ^j .

Recall that the essential norm of a bounded linear operator $T : X \rightarrow Y$ is its distance to the set of compact operators K mapping X into Y , that is,

$$\|T\|_{e, X \rightarrow Y} = \inf\{\|T - K\|_{X \rightarrow Y} : K \text{ is compact}\},$$

where X, Y are Banach spaces and $\|\cdot\|_{X \rightarrow Y}$ is the operator norm.

Throughout this paper, we say that $A \lesssim B$ if there exists a constant C such that $A \leq CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2. ESSENTIAL NORM OF uC_φ

In this section, we give two estimates for the essential norm of the operator $uC_\varphi : A_\alpha^p \rightarrow \mathcal{B}$ and the operator $uC_\varphi : H^p \rightarrow \mathcal{B}$, respectively.

Theorem 2.1. *Let $1 \leq p < \infty$, $\alpha > -1$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ such that $uC_\varphi : A_\alpha^p \rightarrow \mathcal{B}$ is bounded. Then*

$$\|uC_\varphi\|_{e, A_\alpha^p \rightarrow \mathcal{B}} \approx \max\{A, B\} \approx \max\{P, Q\},$$

where

$$\begin{aligned} A &:= \limsup_{|a| \rightarrow 1} \|uC_\varphi(f_a)\|_{\mathcal{B}}, \quad B := \limsup_{|a| \rightarrow 1} \|uC_\varphi(g_a)\|_{\mathcal{B}}, \\ P &:= \limsup_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)|u'(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha)/p}}, \quad Q := \limsup_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)|u(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha+p)/p}}. \end{aligned}$$

Proof. First we prove that

$$\max\{A, B\} \lesssim \|uC_\varphi\|_{e, A_\alpha^p \rightarrow \mathcal{B}}.$$

Let $a \in \mathbb{D}$. Define

$$f_a(z) = \frac{(1 - |a|^2)^{1+(2+\alpha)(1-1/p)}}{(1 - \bar{a}z)^{3+\alpha}}, \quad g_a(z) = \frac{(1 - |a|^2)^{1+(2+\alpha)(1-1/p)+1/p}}{(1 - \bar{a}z)^{3+\alpha+1/p}}, \quad z \in \mathbb{D}.$$

It is easy to check that $f_a, g_a \in A_\alpha^p$ and $\|f_a\|_{A_\alpha^p} \lesssim 1, \|g_a\|_{A_\alpha^p} \lesssim 1$ for all $a \in \mathbb{D}$ and f_a, g_a converges to zero uniformly on compact subsets of \mathbb{D} as $|a| \rightarrow 1$. Thus, for any compact operator $K : A_\alpha^p \rightarrow \mathcal{B}$, by Lemma 3.7 of [20] we have

$$\lim_{|a| \rightarrow 1} \|Kf_a\|_{\mathcal{B}} = 0, \quad \lim_{|a| \rightarrow 1} \|Kg_a\|_{\mathcal{B}} = 0.$$

Hence

$$\|uC_\varphi - K\|_{A_\alpha^p \rightarrow \mathcal{B}} \gtrsim \|(uC_\varphi - K)f_a\|_{\mathcal{B}} \geq \|uC_\varphi f_a\|_{\mathcal{B}} - \|Kf_a\|_{\mathcal{B}},$$

and

$$\|uC_\varphi - K\|_{A_\alpha^p \rightarrow \mathcal{B}} \gtrsim \|(uC_\varphi - K)g_a\|_{\mathcal{B}} \geq \|uC_\varphi g_a\|_{\mathcal{B}} - \|Kg_a\|_{\mathcal{B}}.$$

Taking $\limsup_{|a| \rightarrow 1}$ to the last two inequalities on both sides, we obtain

$$\|uC_\varphi - K\|_{A_\alpha^p \rightarrow \mathcal{B}} \gtrsim A, \quad \|uC_\varphi - K\|_{A_\alpha^p \rightarrow \mathcal{B}} \gtrsim B.$$

Therefore, by the definition of the essential norm, we get

$$\|uC_\varphi\|_{e, A_\alpha^p \rightarrow \mathcal{B}} = \inf_K \|uC_\varphi - K\|_{A_\alpha^p \rightarrow \mathcal{B}} \gtrsim \max\{A, B\}.$$

Next, set

$$h_a(z) = f_a - g_a, \quad k_a(z) = f_a - \frac{3 + \alpha}{3 + \alpha + 1/p} g_a.$$

It is also easy to check that $h_a, k_a \in A_\alpha^p$ and $\|h_a\|_{A_\alpha^p} \lesssim 1, \|k_a\|_{A_\alpha^p} \lesssim 1$ for all $a \in \mathbb{D}$ and h_a, k_a converges to zero uniformly on compact subsets of \mathbb{D} as $|a| \rightarrow 1$. Hence, for any $b_j \in \mathbb{D}$ such that $|\varphi(b_j)| \rightarrow 1$ and any compact operator $K : A_\alpha^p \rightarrow \mathcal{B}$, we have

$$\|uC_\varphi - K\|_{A_\alpha^p \rightarrow \mathcal{B}} \gtrsim \|(uC_\varphi - K)h_{\varphi(b_j)}\|_{\mathcal{B}} \geq \|uC_\varphi h_{\varphi(b_j)}\|_{\mathcal{B}} - \|Kh_{\varphi(b_j)}\|_{\mathcal{B}},$$

and

$$\|uC_\varphi - K\|_{A_\alpha^p \rightarrow \mathcal{B}} \gtrsim \|(uC_\varphi - K)k_{\varphi(b_j)}\|_{\mathcal{B}} \geq \|uC_\varphi k_{\varphi(b_j)}\|_{\mathcal{B}} - \|Kk_{\varphi(b_j)}\|_{\mathcal{B}}.$$

Taking $\limsup_{|\varphi(b_j)| \rightarrow 1}$ to the last two inequalities on both sides we get

$$\|uC_\varphi - K\|_{A_\alpha^p \rightarrow \mathcal{B}} \gtrsim \limsup_{|\varphi(b_j)| \rightarrow 1} \|uC_\varphi h_{\varphi(b_j)}\|_{\mathcal{B}} \gtrsim \limsup_{|\varphi(b_j)| \rightarrow 1} \frac{(1 - |b_j|^2)|u'(b_j)|}{(1 - |\varphi(b_j)|^2)^{(2+\alpha)/p}} = P,$$

and

$$\|uC_\varphi - K\|_{A_\alpha^p \rightarrow \mathcal{B}} \gtrsim \limsup_{|\varphi(b_j)| \rightarrow 1} \|uC_\varphi k_{\varphi(b_j)}\|_{\mathcal{B}} \gtrsim \limsup_{|\varphi(b_j)| \rightarrow 1} \frac{(1 - |b_j|^2)|u(b_j)\varphi'(b_j)|}{(1 - |\varphi(b_j)|^2)^{(2+\alpha+p)/p}} = Q.$$

By the definition of the essential norm, we obtain

$$\|uC_\varphi\|_{e, A_\alpha^p \rightarrow \mathcal{B}} = \inf_K \|uC_\varphi - K\|_{A_\alpha^p \rightarrow \mathcal{B}} \gtrsim \max\{P, Q\}.$$

Finally, we prove that

$$\|uC_\varphi\|_{e, A_\alpha^p \rightarrow \mathcal{B}} \lesssim \max\{A, B\}, \text{ and } \|uC_\varphi\|_{e, A_\alpha^p \rightarrow \mathcal{B}} \lesssim \max\{P, Q\}.$$

For $r \in [0, 1)$, set $K_r : H(\mathbb{D}) \rightarrow H(\mathbb{D})$ by

$$(K_r f)(z) = f_r(z) = f(rz), \quad f \in H(\mathbb{D}).$$

It is clear that K_r is compact on A_α^p and $\|K_r\|_{A_\alpha^p \rightarrow A_\alpha^p} \leq 1$. Let $\{r_j\} \subset (0, 1)$ be a sequence such that $r_j \rightarrow 1$ as $j \rightarrow \infty$. Then for all positive integer j , the operator $uC_\varphi K_{r_j} : A_\alpha^p \rightarrow \mathcal{B}$ is compact. By the definition of the essential norm we have

$$(2.1) \quad \|uC_\varphi\|_{e, A_\alpha^p \rightarrow \mathcal{B}} \leq \limsup_{j \rightarrow \infty} \|uC_\varphi - uC_\varphi K_{r_j}\|_{A_\alpha^p \rightarrow \mathcal{B}}.$$

Thus, we only need to show that

$$(2.2) \quad \limsup_{j \rightarrow \infty} \|uC_\varphi - uC_\varphi K_{r_j}\|_{A_\alpha^p \rightarrow \mathcal{B}} \lesssim \max\{A, B\},$$

and

$$(2.3) \quad \limsup_{j \rightarrow \infty} \|uC_\varphi - uC_\varphi K_{r_j}\|_{A_\alpha^p \rightarrow \mathcal{B}} \lesssim \max\{P, Q\}.$$

For any $f \in A_\alpha^p$ such that $\|f\|_{A_\alpha^p} \leq 1$, we consider

$$\|(uC_\varphi - uC_\varphi K_{r_j})f\|_{\mathcal{B}} = |u(0)f(\varphi(0)) - u(0)f(r_j\varphi(0))| + \|u \cdot (f - f_{r_j}) \circ \varphi\|_{\mathcal{B}}.$$

It is clear that $\lim_{j \rightarrow \infty} |u(0)f(\varphi(0)) - u(0)f(r_j\varphi(0))| = 0$. Now we estimate

$$\begin{aligned}
& \limsup_{j \rightarrow \infty} \|u \cdot (f - f_{r_j}) \circ \varphi\|_{\mathcal{B}} \\
&= \limsup_{j \rightarrow \infty} \sup_{|\varphi(z)| \leq r_N} (1 - |z|^2) |(f - f_{r_j})'(\varphi(z))| |\varphi'(z)| |u(z)| \\
&\quad + \limsup_{j \rightarrow \infty} \sup_{|\varphi(z)| > r_N} (1 - |z|^2) |(f - f_{r_j})'(\varphi(z))| |\varphi'(z)| |u(z)| \\
&\quad + \limsup_{j \rightarrow \infty} \sup_{|\varphi(z)| \leq r_N} (1 - |z|^2) |(f - f_{r_j})(\varphi(z))| |u'(z)| \\
&\quad + \limsup_{j \rightarrow \infty} \sup_{|\varphi(z)| > r_N} (1 - |z|^2) |(f - f_{r_j})(\varphi(z))| |u'(z)| \\
(2.4) \quad &= Q_1 + Q_2 + Q_3 + Q_4,
\end{aligned}$$

where $N \in \mathbb{N}$ is large enough such that $r_j \geq \frac{1}{2}$ for all $j \geq N$,

$$Q_1 := \limsup_{j \rightarrow \infty} \sup_{|\varphi(z)| \leq r_N} (1 - |z|^2) |(f - f_{r_j})'(\varphi(z))| |\varphi'(z)| |u(z)|,$$

$$Q_2 := \limsup_{j \rightarrow \infty} \sup_{|\varphi(z)| > r_N} (1 - |z|^2) |(f - f_{r_j})'(\varphi(z))| |\varphi'(z)| |u(z)|,$$

$$Q_3 := \limsup_{j \rightarrow \infty} \sup_{|\varphi(z)| \leq r_N} (1 - |z|^2) |(f - f_{r_j})(\varphi(z))| |u'(z)|,$$

and

$$Q_4 := \limsup_{j \rightarrow \infty} \sup_{|\varphi(z)| > r_N} (1 - |z|^2) |(f - f_{r_j})(\varphi(z))| |u'(z)|.$$

Since $uC_\varphi : A_\alpha^p \rightarrow \mathcal{B}$ is bounded, applying the operator uC_φ to 1 and z , we easily get that $u \in \mathcal{B}$ and

$$\widetilde{K} := \sup_{z \in \mathbb{D}} (1 - |z|^2) |\varphi'(z)| |u(z)| < \infty.$$

Since $r_j f'_{r_j} \rightarrow f'$ uniformly on compact subsets of \mathbb{D} as $j \rightarrow \infty$, we have

$$(2.5) \quad Q_1 \leq \widetilde{K} \limsup_{j \rightarrow \infty} \sup_{|w| \leq r_N} |f'(w) - r_j f'(r_j w)| = 0.$$

Also, from the fact that $u \in \mathcal{B}$ and $f_{r_j} \rightarrow f$ uniformly on compact subsets of \mathbb{D} as $j \rightarrow \infty$, we have

$$(2.6) \quad Q_3 \leq \|u\|_{\mathcal{B}} \limsup_{j \rightarrow \infty} \sup_{|w| \leq r_N} |f(w) - f(r_j w)| = 0.$$

Next we consider Q_2 . We have $Q_2 \leq \limsup_{j \rightarrow \infty} (S_1 + S_2)$, where

$$S_1 := \sup_{|\varphi(z)| > r_N} (1 - |z|^2) |f'(\varphi(z))| |\varphi'(z)| |u(z)|$$

and

$$S_2 := \sup_{|\varphi(z)| > r_N} (1 - |z|^2) r_j |f'(r_j \varphi(z))| |\varphi'(z)| |u(z)|.$$

First we estimate S_1 . Using the fact that $\|f\|_{A_\alpha^p} \leq 1$, we have

$$\begin{aligned}
 S_1 &= \sup_{|\varphi(z)| > r_N} (1 - |z|^2) |f'(\varphi(z))| |\varphi'(z)| |u(z)| \\
 &\lesssim \frac{1}{r_N} \|f\|_{A_\alpha^p} \sup_{|\varphi(z)| > r_N} (1 - |z|^2) |\varphi'(z)| |u(z)| \frac{|\varphi(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha+p)/p}} \\
 &\lesssim \frac{1}{p} \sup_{|\varphi(z)| > r_N} \sup_{|a| > r_N} (1 - |z|^2) |\varphi'(z)| |u(z)| \times \frac{|\varphi(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha+p)/p}} \\
 &\lesssim \sup_{|a| > r_N} \|uC_\varphi(f_a - g_a)\|_{\mathcal{B}} \\
 &\lesssim \sup_{|a| > r_N} \|uC_\varphi f_a\|_{\mathcal{B}} + \sup_{|a| > r_N} \|uC_\varphi g_a\|_{\mathcal{B}}.
 \end{aligned}$$

Taking limit as $N \rightarrow \infty$ we obtain

$$\begin{aligned}
 \limsup_{j \rightarrow \infty} S_1 &\lesssim \limsup_{|a| \rightarrow 1} \frac{(1 - |z|^2) |\varphi'(z)| |u(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha+p)/p}} = Q \\
 &\lesssim \limsup_{|a| \rightarrow 1} \|uC_\varphi f_a\|_{\mathcal{B}} + \limsup_{|a| \rightarrow 1} \|uC_\varphi g_a\|_{\mathcal{B}}.
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 \limsup_{j \rightarrow \infty} S_2 &\lesssim \limsup_{|a| \rightarrow 1} \frac{(1 - |z|^2) |\varphi'(z)| |u(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha+p)/p}} = Q \\
 &\lesssim \limsup_{|a| \rightarrow 1} \|uC_\varphi f_a\|_{\mathcal{B}} + \limsup_{|a| \rightarrow 1} \|uC_\varphi g_a\|_{\mathcal{B}}.
 \end{aligned}$$

i.e., we get that

$$(2.7) \quad Q_2 \lesssim Q \lesssim A + B \lesssim \max\{A, B\}.$$

Next we consider Q_4 . We have $Q_4 \leq \limsup_{j \rightarrow \infty} (S_3 + S_4)$, where

$$S_3 := \sup_{|\varphi(z)| > r_N} (1 - |z|^2) |f(\varphi(z))| |u'(z)|, S_4 := \sup_{|\varphi(z)| > r_N} (1 - |z|^2) |f(r_j \varphi(z))| |u'(z)|.$$

Similarly, we have

$$\begin{aligned}
 S_3 &\lesssim \sup_{|\varphi(z)| > r_N} \sup_{|a| > r_N} (1 - |z|^2) |u'(z)| \frac{1}{(1 - |\varphi(z)|^2)^{(2+\alpha)/p}} \\
 &\lesssim \sup_{|a| > r_N} \left\| uC_\varphi f_a - \frac{3 + \alpha}{3 + \alpha + 1/p} uC_\varphi g_a \right\|_{\mathcal{B}} \\
 &\leq \sup_{|a| > r_N} \|uC_\varphi f_a\|_{\mathcal{B}} + \frac{3 + \alpha}{3 + \alpha + 1/p} \sup_{|a| > r_N} \|uC_\varphi g_a\|_{\mathcal{B}} \\
 &\leq \sup_{|a| > r_N} \|uC_\varphi f_a\|_{\mathcal{B}} + \sup_{|a| > r_N} \|uC_\varphi g_a\|_{\mathcal{B}}.
 \end{aligned}$$

Taking limit as $N \rightarrow \infty$ we obtain

$$\begin{aligned} \limsup_{j \rightarrow \infty} S_3 &\lesssim \limsup_{|a| \rightarrow 1} \frac{(1 - |z|^2)|u'(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha)/p}} = P \\ &\lesssim \limsup_{|a| \rightarrow 1} \|uC_\varphi f_a\|_{\mathcal{B}} + \limsup_{|a| \rightarrow 1} \|uC_\varphi g_a\|_{\mathcal{B}} = A + B. \end{aligned}$$

Similarly, we have $\limsup_{j \rightarrow \infty} S_4 \lesssim P \lesssim A + B$, i.e., we get that

$$(2.8) \quad Q_4 \lesssim P \lesssim A + B.$$

Hence, by (2.4), (2.5), (2.6), (2.7) and (2.8) we get

$$\begin{aligned} \limsup_{j \rightarrow \infty} \|uC_\varphi - uC_\varphi K_{r_j}\|_{A_\alpha^p \rightarrow \mathcal{B}} &= \limsup_{j \rightarrow \infty} \sup_{\|f\|_{A_\alpha^p} \leq 1} \|(uC_\varphi - uC_\varphi K_{r_j})f\|_{\mathcal{B}} \\ &= \limsup_{j \rightarrow \infty} \sup_{\|f\|_{A_\alpha^p} \leq 1} \|u \cdot (f - f_{r_j}) \circ \varphi\|_{\mathcal{B}} \\ (2.9) \quad &\lesssim P + Q \lesssim A + B. \end{aligned}$$

Therefore, by (2.1) and (2.9), we obtain

$$\|uC_\varphi\|_{e, A_\alpha^p \rightarrow \mathcal{B}} \lesssim P + Q \lesssim \max\{P, Q\}$$

and

$$\|uC_\varphi\|_{e, A_\alpha^p \rightarrow \mathcal{B}} \lesssim A + B \lesssim \max\{A, B\}.$$

This completes the proof of the theorem. \square

The Hardy space H^p can be viewed as limiting space of A_α^p as α decreases to -1 . Hence, from Theorem 2.1, we get the following result.

Theorem 2.2. *Let $1 \leq p < \infty$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ such that $uC_\varphi : H^p \rightarrow \mathcal{B}$ is bounded. Then*

$$\begin{aligned} \|uC_\varphi\|_{e, H^p \rightarrow \mathcal{B}} &\approx \max \left\{ \limsup_{|a| \rightarrow 1} \|uC_\varphi(p_a)\|_{\mathcal{B}}, \limsup_{|a| \rightarrow 1} \|uC_\varphi(q_a)\|_{\mathcal{B}} \right\} \\ &\approx \max \left\{ \limsup_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)|u'(z)|}{(1 - |\varphi(z)|^2)^{1/p}}, \limsup_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)|u(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{(1+p)/p}} \right\}. \end{aligned}$$

From Theorems 2.1 and 2.2, we immediately get the following two corollaries.

Corollary 2.1. *Let $1 \leq p < \infty$, $\alpha > -1$ and $\varphi \in S(\mathbb{D})$ such that $C_\varphi : A_\alpha^p \rightarrow \mathcal{B}$ is bounded. Then*

$$\begin{aligned} \|C_\varphi\|_{e, A_\alpha^p \rightarrow \mathcal{B}} &\approx \limsup_{|a| \rightarrow 1} \|C_\varphi(f_a)\|_{\mathcal{B}} \approx \limsup_{|a| \rightarrow 1} \|C_\varphi(g_a)\|_{\mathcal{B}} \\ &\approx \limsup_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)|\varphi'(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha+p)/p}}. \end{aligned}$$

Corollary 2.2. *Let $1 \leq p < \infty$ and $\varphi \in S(\mathbb{D})$ such that $C_\varphi : H^p \rightarrow \mathcal{B}$ is bounded. Then*

$$\begin{aligned} \|C_\varphi\|_{e, H^p \rightarrow \mathcal{B}} &\approx \limsup_{|a| \rightarrow 1} \|C_\varphi(p_a)\|_{\mathcal{B}} \approx \limsup_{|a| \rightarrow 1} \|C_\varphi(q_a)\|_{\mathcal{B}} \\ &\approx \limsup_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)|\varphi'(z)|}{(1 - |\varphi(z)|^2)^{(1+p)/p}}. \end{aligned}$$

3. NEW CHARACTERIZATION OF uC_φ

In this section, motivated by [4], we give a new characterization for the boundedness, compactness and essential norm for the weighted composition operators $uC_\varphi : A_\alpha^p \rightarrow \mathcal{B}$ and $uC_\varphi : H^p \rightarrow \mathcal{B}$. For this purpose, we state some lemmas which will be used.

Lemma 3.1. [15] *Let v and w be radial, non-increasing weights tending to zero at the boundary of \mathbb{D} . Then the following statements hold.*

(a) *The weighted composition operator $uC_\varphi : H_v^\infty \rightarrow H_w^\infty$ is bounded if and only if*

$$\sup_{z \in \mathbb{D}} \frac{w(z)}{\widetilde{v}(\varphi(z))} |\varphi(z)| < \infty.$$

Moreover, the following holds

$$\|uC_\varphi\|_{H_v^\infty \rightarrow H_w^\infty} = \sup_{z \in \mathbb{D}} \frac{w(z)}{\widetilde{v}(\varphi(z))} |\varphi(z)|.$$

(b) *Suppose $uC_\varphi : H_v^\infty \rightarrow H_w^\infty$ is bounded. Then*

$$\|uC_\varphi\|_{e, H_v^\infty \rightarrow H_w^\infty} = \lim_{s \rightarrow 1^-} \sup_{|\varphi(z)| > s} \frac{w(z)}{\widetilde{v}(\varphi(z))} |\varphi(z)|.$$

Lemma 3.2. [6] *Let v and w be radial, non-increasing weights tending to zero at the boundary of \mathbb{D} . Then the following statements hold.*

(a) *$uC_\varphi : H_v^\infty \rightarrow H_w^\infty$ is bounded if and only if*

$$\sup_{k \geq 0} \frac{\|u\varphi^k\|_w}{\|z^k\|_v} < \infty,$$

with the norm comparable to the above supremum.

(b) *Suppose $uC_\varphi : H_v^\infty \rightarrow H_w^\infty$ is bounded. Then*

$$\|uC_\varphi\|_{e, H_v^\infty \rightarrow H_w^\infty} = \limsup_{k \rightarrow \infty} \frac{\|u\varphi^k\|_w}{\|z^k\|_v}.$$

Lemma 3.3.[5] *For $\alpha > 0$, we have $\lim_{k \rightarrow \infty} k^\alpha \|z^{k-1}\|_{v_\alpha} = (\frac{2\alpha}{e})^\alpha$.*

Theorem 3.1. *Let $1 \leq p < \infty$, $\alpha > -1$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then the operator $uC_\varphi : A_\alpha^p \rightarrow \mathcal{B}$ is bounded if and only if*

$$(3.1) \quad \sup_{j \geq 1} j^{(2+\alpha)/p} \|I_u(\varphi^j)\|_{\mathcal{B}} < \infty \text{ and } \sup_{j \geq 1} j^{(2+\alpha)/p} \|J_u(\varphi^{j-1})\|_{\mathcal{B}} < \infty,$$

where

$$I_u g(z) = \int_0^z g'(\xi) u(\xi) d\xi, \quad J_u g(z) = \int_0^z g(\xi) u'(\xi) d\xi, \quad z \in \mathbb{D}, g \in H(\mathbb{D}).$$

Proof. By Theorem A, $uC_\varphi : A_\alpha^p \rightarrow \mathcal{B}$ is bounded if and only if

$$(3.2) \quad \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)|u'(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha)/p}} < \infty \text{ and } \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)|u(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha+p)/p}} < \infty,$$

which are equivalent to the weighted composition operator $u'C_\varphi : H_{v_{(2+\alpha)/p}}^\infty \rightarrow H_{v_1}^\infty$ is bounded and $u\varphi'C_\varphi : H_{v_{(2+\alpha+p)/p}}^\infty \rightarrow H_{v_1}^\infty$ is bounded, respectively. By Lemma 3.2, we see that two inequalities in (3.2) are equivalent to

$$\sup_{j \geq 1} \frac{\|u'\varphi^{j-1}\|_{v_1}}{\|z^{j-1}\|_{v_{(2+\alpha)/p}}} < \infty \text{ and } \sup_{j \geq 1} \frac{\|u\varphi'\varphi^{j-1}\|_{v_1}}{\|z^{j-1}\|_{v_{(2+\alpha+p)/p}}} < \infty,$$

respectively. Since $I_u f(0) = 0, J_u f(0) = 0$,

$$(I_u(\varphi^j)(z))' = ju(z)\varphi'(z)\varphi^{j-1}(z), \quad (J_u(\varphi^{j-1})(z))' = u'(z)\varphi^{j-1}(z),$$

by Lemma 3.3, we see that $uC_\varphi : A_\alpha^p \rightarrow \mathcal{B}$ is bounded if and only if

$$(3.3) \quad \begin{aligned} \sup_{j \geq 1} j^{(2+\alpha)/p} \|J_u(\varphi^{j-1})\|_{\mathcal{B}} &= \sup_{j \geq 1} j^{(2+\alpha)/p} \|u'\varphi^{j-1}\|_{v_1} \\ &\approx \sup_{j \geq 1} \frac{j^{(2+\alpha)/p} \|u'\varphi^{j-1}\|_{v_1}}{j^{(2+\alpha)/p} \|z^{j-1}\|_{v_{(2+\alpha)/p}}} < \infty \end{aligned}$$

and

$$(3.4) \quad \begin{aligned} \sup_{j \geq 1} j^{(2+\alpha)/p} \|I_u(\varphi^j)\|_{\mathcal{B}} &= \sup_{j \geq 1} j^{(2+\alpha+p)/p} \|u\varphi'\varphi^{j-1}\|_{v_1} \\ &\approx \sup_{j \geq 1} \frac{j^{(2+\alpha+p)/p} \|u\varphi'\varphi^{j-1}\|_{v_1}}{j^{(2+\alpha+p)/p} \|z^{j-1}\|_{v_{(2+\alpha+p)/p}}} < \infty. \end{aligned}$$

The proof is complete. \square

Theorem 3.2. *Let $1 \leq p < \infty$, $\alpha > -1$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ such that the operator $uC_\varphi : A_\alpha^p \rightarrow \mathcal{B}$ is bounded. Then*

$$\|uC_\varphi\|_{\ell, A_\alpha^p \rightarrow \mathcal{B}} \approx \max \left\{ \limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|I_u(\varphi^j)\|_{\mathcal{B}}, \limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|J_u(\varphi^{j-1})\|_{\mathcal{B}} \right\}.$$

Proof. By Theorem A and Lemma 3.1, $uC_\varphi : A_\alpha^p \rightarrow \mathcal{B}$ is bounded if and only if the weighted composition operator $u'C_\varphi : H_{v(2+\alpha)/p}^\infty \rightarrow H_{v_1}^\infty$ is bounded and $u\varphi'C_\varphi : H_{v(2+\alpha+p)/p}^\infty \rightarrow H_{v_1}^\infty$ is bounded. By Lemmas 3.2 and 3.3, we get

$$\begin{aligned}
 \|u'C_\varphi\|_{e, H_{v(2+\alpha)/p}^\infty \rightarrow H_{v_1}^\infty} &= \limsup_{j \rightarrow \infty} \frac{\|u'\varphi^{j-1}\|_{v_1}}{\|z^{j-1}\|_{v(2+\alpha)/p}} = \limsup_{j \rightarrow \infty} \frac{j^{(2+\alpha)/p} \|u'\varphi^{j-1}\|_{v_1}}{j^{(2+\alpha)/p} \|z^{j-1}\|_{v(2+\alpha)/p}} \\
 &\approx \limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|u'\varphi^{j-1}\|_{v_1} \\
 (3.5) \quad &= \limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|J_u(\varphi^{j-1})\|_{\mathcal{B}}
 \end{aligned}$$

and

$$\begin{aligned}
 \|u\varphi'C_\varphi\|_{e, H_{v(2+\alpha+p)/p}^\infty \rightarrow H_{v_1}^\infty} &= \limsup_{j \rightarrow \infty} \frac{\|u\varphi'\varphi^{j-1}\|_{v_1}}{\|z^{j-1}\|_{v(2+\alpha+p)/p}} \\
 &\approx \limsup_{j \rightarrow \infty} j^{(2+\alpha+p)/p} \|u\varphi'\varphi^{j-1}\|_{v_1} \\
 (3.6) \quad &= \limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|I_u(\varphi^j)\|_{\mathcal{B}}.
 \end{aligned}$$

The upper estimate. From the fact $(uC_\varphi f)'(z) = u'(z)f(\varphi(z)) + u(z)\varphi'(z)f'(\varphi(z))$, it is easy to see that

$$(3.7) \quad \|uC_\varphi\|_{e, A_\alpha^p \rightarrow \mathcal{B}} \leq \|u'C_\varphi\|_{e, H_{v(2+\alpha)/p}^\infty \rightarrow H_{v_1}^\infty} + \|u\varphi'C_\varphi\|_{e, H_{v(2+\alpha+p)/p}^\infty \rightarrow H_{v_1}^\infty}.$$

Then, by (3.5), (3.6) and (3.7) we get

$$\begin{aligned}
 \|uC_\varphi\|_{e, A_\alpha^p \rightarrow \mathcal{B}} &\lesssim \limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|I_u(\varphi^j)\|_{\mathcal{B}} + \limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|J_u(\varphi^{j-1})\|_{\mathcal{B}} \\
 &\lesssim \max \left\{ \limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|I_u(\varphi^j)\|_{\mathcal{B}}, \limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|J_u(\varphi^{j-1})\|_{\mathcal{B}} \right\}.
 \end{aligned}$$

The lower estimate. From Theorem 2.1 and Lemma 3.1, we have

$$\|uC_\varphi\|_{e, A_\alpha^p \rightarrow \mathcal{B}} \gtrsim P = \|u'C_\varphi\|_{e, H_{(2+\alpha)/p}^\infty \rightarrow H_{v_1}^\infty} \approx \limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|J_u(\varphi^{j-1})\|_{\mathcal{B}}$$

and

$$\|uC_\varphi\|_{e, A_\alpha^p \rightarrow \mathcal{B}} \gtrsim Q = \|u\varphi'C_\varphi\|_{e, H_{v(2+\alpha+p)/p}^\infty \rightarrow H_{v_1}^\infty} \approx \limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|I_u(\varphi^j)\|_{\mathcal{B}}.$$

Therefore,

$$\|uC_\varphi\|_{e, A_\alpha^p \rightarrow \mathcal{B}} \gtrsim \max \left\{ \limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|I_u(\varphi^j)\|_{\mathcal{B}}, \limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|J_u(\varphi^{j-1})\|_{\mathcal{B}} \right\}.$$

This completes the proof. \square

From Theorem 3.2, we immediately get the following result.

Theorem 3.3. Let $1 \leq p < \infty$, $\alpha > -1$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ such that $uC_\varphi : A_\alpha^p \rightarrow \mathcal{B}$ is bounded. Then the operator $uC_\varphi : A_\alpha^p \rightarrow \mathcal{B}$ is compact if and only if

$$\limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|I_u(\varphi^j)\|_{\mathcal{B}} = 0 \text{ and } \limsup_{j \rightarrow \infty} j^{(2+\alpha)/p} \|J_u(\varphi^{j-1})\|_{\mathcal{B}} = 0.$$

We end this section with a new characterization of boundedness, compactness and essential norm of the operator $uC_\varphi : H^p \rightarrow \mathcal{B}$, which follows from Theorems 3.1, 3.2 and 3.3 by taking the limit as α decreases to -1 .

Theorem 3.4. Let $1 \leq p < \infty$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then the following statements hold.

(a) The operator $uC_\varphi : H^p \rightarrow \mathcal{B}$ is bounded if and only if

$$\sup_{j \geq 1} j^{1/p} \|I_u(\varphi^j)\|_{\mathcal{B}} < \infty \text{ and } \sup_{j \geq 1} j^{1/p} \|J_u(\varphi^{j-1})\|_{\mathcal{B}} < \infty.$$

(b) If the operator $uC_\varphi : H^p \rightarrow \mathcal{B}$ is bounded. Then $uC_\varphi : H^p \rightarrow \mathcal{B}$ is compact if and only if

$$\limsup_{j \rightarrow \infty} j^{1/p} \|I_u(\varphi^j)\|_{\mathcal{B}} = 0 \text{ and } \limsup_{j \rightarrow \infty} j^{1/p} \|J_u(\varphi^{j-1})\|_{\mathcal{B}} = 0.$$

Moreover

$$\|uC_\varphi\|_{e, H^p \rightarrow \mathcal{B}} \approx \max \left\{ \limsup_{j \rightarrow \infty} j^{1/p} \|I_u(\varphi^j)\|_{\mathcal{B}}, \limsup_{j \rightarrow \infty} j^{1/p} \|J_u(\varphi^{j-1})\|_{\mathcal{B}} \right\}.$$

From the above results, we immediately get the following new characterization of the operator $C_\varphi : A_\alpha^p(\text{or } H^p) \rightarrow \mathcal{B}$.

Corollary 3.1. Let $1 \leq p < \infty$, $\alpha > -1$ and $\varphi \in S(\mathbb{D})$. Then the following statements hold.

(a) The operator $C_\varphi : A_\alpha^p \rightarrow \mathcal{B}$ is bounded if and only if $\sup_{j \geq 1} j^{(\alpha+2)/p} \|\varphi^j\|_{\mathcal{B}} < \infty$.

(b) If the operator $C_\varphi : A_\alpha^p \rightarrow \mathcal{B}$ is bounded. Then $C_\varphi : A_\alpha^p \rightarrow \mathcal{B}$ is compact if and only if $\limsup_{j \rightarrow \infty} j^{(\alpha+2)/p} \|\varphi^j\|_{\mathcal{B}} = 0$. Moreover,

$$\|C_\varphi\|_{e, A_\alpha^p \rightarrow \mathcal{B}} \approx \limsup_{j \rightarrow \infty} j^{(\alpha+2)/p} \|\varphi^j\|_{\mathcal{B}}.$$

Corollary 3.2. Let $1 \leq p < \infty$ and $\varphi \in S(\mathbb{D})$. Then the following statements hold.

(a) The operator $C_\varphi : H^p \rightarrow \mathcal{B}$ is bounded if and only if $\sup_{j \geq 1} j^{1/p} \|\varphi^j\|_{\mathcal{B}} < \infty$.

(b) If the operator $C_\varphi : H^p \rightarrow \mathcal{B}$ is bounded, then $C_\varphi : H^p \rightarrow \mathcal{B}$ is compact if and only if $\limsup_{j \rightarrow \infty} j^{1/p} \|\varphi^j\|_{\mathcal{B}} = 0$. Moreover

$$\|C_\varphi\|_{e, H^p \rightarrow \mathcal{B}} \approx \limsup_{j \rightarrow \infty} j^{1/p} \|\varphi^j\|_{\mathcal{B}}.$$

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SONGXIAO LI, INSTITUTE OF SYSTEMS ENGINEERING, MACAU UNIVERSITY OF SCIENCE AND TECHNOLOGY, AVENIDA WAI LONG, TAIPA, MACAU.

E-mail address: jyulsx@163.com

RUI SHEN QIAN, SCHOOL OF MATHEMATICS AND COMPUTATION SCIENCE, LINGNAN NORMAL UNIVERSITY, ZHANJIANG 524048, GUANGDONG, P. R. CHINA

E-mail address: qianruishen@sina.cn

JIZHEN ZHOU, SCHOOL OF SCIENCES, ANHUI UNIVERSITY OF SCIENCE AND TECHNOLOGY, HUAINAN, ANHUI 232001, P. R. CHINA

E-mail address: hope189@163.com